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FAST TRACK COMMUNICATION

Generalized measurement of the non-normal two-boson operator $Z_\gamma = a_1 + \gamma a_2^\dagger$

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Abstract
We address the generalized measurement of the two-boson operator $Z_\gamma = a_1 + \gamma a_2^\dagger$ which, for $|\gamma|^2 \neq 1$, is not normal and cannot be detected by a joint measurement of quadratures on the two bosons. We explicitly construct the minimal Naimark extension, which involves a single additional bosonic system, and present its decomposition in terms of two-boson linear $SU(2)$ interactions. The statistics of the measurement and the added noise are analysed in detail. Results are exploited to revisit the Caves–Shapiro concept of generalized phase observable based on heterodyne detection.

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(Some figures in this article are in colour only in the electronic version)

The two-boson operator

$$Z_\gamma = a_1 + \gamma a_2^\dagger \tag{1}$$

is normal $\left[Z_\gamma, Z_\gamma^\dagger\right] = 1 - |\gamma|^2$ for $|\gamma| = 1$. In this case, the real $X_\gamma = \frac{1}{2}\left(Z_\gamma + Z_\gamma^\dagger\right)$ and the imaginary $Y_\gamma = \frac{\mathrm{i}}{2}\left(Z_\gamma - Z_\gamma^\dagger\right)$ parts of Z_γ commute $[X_\gamma, Y_\gamma] = 0$ and can be jointly measured. Actually they correspond to canonical sum- and difference-quadratures of the two modes e.g. for $\gamma = \pm 1$

$$X_\gamma = \frac{1}{\sqrt{2}}(q_1 \pm q_2), \qquad Y_\gamma = \frac{1}{\sqrt{2}}(p_1 \mp p_2), \qquad \gamma = \pm 1 \tag{2}$$

where, for $k = 1, 2$,

$$q_k = \frac{1}{\sqrt{2}}\left(a_k^\dagger + a_k\right) \qquad p_k = \frac{\mathrm{i}}{\sqrt{2}}\left(a_k^\dagger - a_k\right) \quad [q_j, p_k] = \mathrm{i}\delta_{jk}. \tag{3}$$

On the other hand, for $|\gamma| \neq 1$, we have

$$X_\gamma = \frac{1}{\sqrt{2}}\left(q_1 + |\gamma| x_{2,\theta_\gamma}\right) \qquad Y_\gamma = \frac{1}{\sqrt{2}}\left(p_1 - |\gamma| x_{2,\theta_\gamma+\pi/2}\right), \tag{4}$$

where $x_{k,\phi} = \frac{1}{\sqrt{2}}\left(a_k^\dagger \mathrm{e}^{\mathrm{i}\phi} + a_k\,\mathrm{e}^{-\mathrm{i}\phi}\right)$ is a rotated quadrature of the kth boson and $\theta_\gamma = \arg\gamma$. In this case, the two operators do no commute $[X_\gamma, Y_\gamma] = \frac{\mathrm{i}}{2}(1-|\gamma|^2)$ and a generalized measurement should be devised. Indeed, the eigenstates of Z_γ for $\gamma \neq 1$,

$$|z\rangle\rangle_\gamma = D(z) \otimes \mathbb{I}|\gamma\rangle\rangle,$$

where $D(z) = \exp\left\{za_1^\dagger - z^* a_1\right\}$ is the displacement operator and $|\gamma\rangle\rangle = \sqrt{1-|\gamma|^2}\sum_n \gamma^n |n\rangle \otimes |n\rangle$, *do not* provide a resolution of the identity. We have

$$\int \frac{\mathrm{d}^2 z}{\pi} |z\rangle\rangle_{\gamma\,\gamma}\langle\langle z| = (1-|\gamma|^2)|\gamma|^{2a^\dagger a}.$$

We first note that $Z_\gamma = R_{\theta_\gamma}^\dagger Z_{|\gamma|} R_{\theta_\gamma}$, where $R_\phi = \exp\left(\mathrm{i}\phi a_2^\dagger a_2\right)$ and therefore, without loss of generality, we may restrict attention to the case of real positive γ. In this case, we have

$$X_\gamma = \frac{1}{\sqrt{2}}(q_1 + \gamma q_2) \qquad Y_\gamma = \frac{1}{\sqrt{2}}(p_1 - \gamma p_2). \tag{5}$$

In addition, we note that, up to a permutation of the mode labels, $Z_\gamma = \gamma Z_{\gamma^{-1}}^\dagger$ and therefore, since the multiplicative constant does not influence the measurement scheme, we may further restrict attention to the case $0 < \gamma < 1$.

The operator Z_γ is defined on the Hilbert–Fock space $\mathcal{H}_{12}$ of two harmonic oscillators. A Naimark extension for the operator Z_γ is a triplet $(\mathcal{H}_a, T_\gamma, \sigma)$, where T_γ is an operator defined on an extended Hilbert space $\mathcal{H}_{12} \otimes \mathcal{H}_a$ and σ is a state (density operator) in $\mathcal{H}_a$, such that for any state $R \in \mathcal{H}_{12}$, we have

$$\begin{aligned} \mathrm{Tr}_{12}[RX_\gamma] &= \mathrm{Tr}_{12a}[R \otimes \sigma \,\mathrm{Re}\, T_\gamma] \\ \mathrm{Tr}_{12}[RY_\gamma] &= \mathrm{Tr}_{12a}[R \otimes \sigma \,\mathrm{Im}\, T_\gamma]. \end{aligned} \tag{6}$$

Equations (6) are usually summarized by saying that the operator T_γ *traces* the operator Z_γ. Of course, equations (6) do not hold for higher moments: the generalized measurement of Z_γ unavoidably introduces some noise of purely quantum origin. In general, we have

$$\begin{aligned} \mathrm{Tr}_{12}\left[RX_\gamma^n\right] &\neq \mathrm{Tr}_{12a}[R \otimes \sigma (\mathrm{Re}\, T_\gamma)^n] \qquad n \geqslant 2 \\ \mathrm{Tr}_{12}\left[RY_\gamma^n\right] &\neq \mathrm{Tr}_{12a}\left[R \otimes \sigma (\mathrm{Im}\, T_\gamma)^n\right] \qquad n \geqslant 2. \end{aligned} \tag{7}$$

In this communication, we look for a minimal Naimark extension, that is an extension involving a single additional bosonic mode a_3. In general, for an operator of the form $T_\gamma = Z_\gamma + f\left(a_3, a_3^\dagger\right)$ the trace condition of equations (6) requires $\mathrm{Tr}_a\left[\sigma f\left(a_3, a_3^\dagger\right)\right] = 0$, whereas the constraint of normality can be written as

$$0 \equiv \left[T_\gamma, T_\gamma^\dagger\right] = \left[Z_\gamma, Z_\gamma^\dagger\right] + \left[f\left(a_3, a_3^\dagger\right), f\left(a_3, a_3^\dagger\right)^\dagger\right]. \tag{8}$$

It is straightforwardly seen that $f\left(a_3, a_3^\dagger\right) = \kappa a_3$ or $f\left(a_3, a_3^\dagger\right) = \kappa a_3^\dagger$, where κ is a real constant, are solutions of equations (6) and (8). In the following, we analyse in detail whether this kind of extensions can be implemented *using only bilinear interactions* among the three modes followed by measurement of quadratures at the output.

The measurement scheme is the following: the modes a_k interact each other via the unitary operator U_γ, which imposes the linear transformation

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = U_\gamma^\dagger \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} U_\gamma = \mathbf{M} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{9}$$

and then, at the output, the quadratures

$$Q_1 = \frac{1}{\sqrt{2}}\left(A_1 + A_1^\dagger\right) \qquad P_2 = \frac{\mathrm{i}}{\sqrt{2}}\left(A_2^\dagger - A_2\right)$$

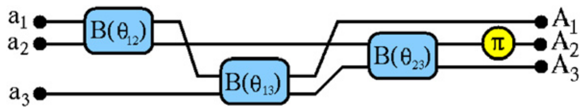

Figure 1. Block diagram of the decomposition of the **M** transformation of equation (12) into three $SU(2)$ transformations, each involving two of the modes, plus a π-rotation. The boxes correspond to evolution operators of the form $B_{jk}(\theta_{jk}) = \mathrm{e}^{-\mathrm{i}\theta_{jk}(a_j a_k^\dagger + a_k a_j^\dagger)}$ (see the text).

are measured with the aim of obtaining, upon the definition $T_\gamma = Q_1 + \mathrm{i}P_2$,

$$\mathrm{Tr}_{12}[RX_\gamma] = \mathrm{Tr}_{12a}[R \otimes \sigma Q_1] \tag{10}$$

$$\mathrm{Tr}_{12}[RY_\gamma] = \mathrm{Tr}_{12a}[R \otimes \sigma P_2] \tag{11}$$

for *any* R, and at least one σ such that $\mathrm{Tr}[\sigma a_3] = 0$. A suitable evolution operator U_γ corresponds to the transformation

$$\mathbf{M} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \gamma & \kappa \\ 1 & -\gamma & -\kappa \\ m_1 & m_2 & m_3 \end{pmatrix}. \tag{12}$$

Upon imposing the constraint of unitarity, i.e. $\left[A_j, A_k^\dagger\right] = \delta_{jk}$, we have the solution

$$\kappa = \sqrt{1-\gamma^2}, \qquad m_1 = 0, \tag{13}$$

$$m_2 = -\sqrt{2(1-\gamma^2)}, \qquad m_3 = \sqrt{2}\gamma, \tag{14}$$

which makes **M** a $U(3)$ transformation and leads to

$$Q_1 = \tfrac{1}{\sqrt{2}}\left(q_1 + \gamma q_2 + \sqrt{1-\gamma^2} q_3\right), \qquad P_2 = \tfrac{1}{\sqrt{2}}\left(p_1 - \gamma p_2 - \sqrt{1-\gamma^2} p_3\right), \tag{15}$$

and, in turn, to $T_\gamma = a_1 + \gamma a_2^\dagger + \kappa a_3^\dagger$. Note that no unitary solution can be found (for $|\gamma| < 1$) for the case $f\left(a_3, a_3^\dagger\right) = \kappa a_3$, i.e. for linear transformation expressing the output modes (A_1, A_2, A_3) as a linear combination of $\left(a_1, a_2, a_3^\dagger\right)$.[1]

A question arises on how the unitary U_γ can be implemented in practice, as for example in a quantum optical setting. As is well known, any $SU(3)$ transformation may be decomposed into a set of $SU(2)$ transformation [6]. In our case, the $U(3)$ **M** transformation may be decomposed using three $SU(2)$ transformations followed by a π-rotation. In figure 1, we report the explicit decomposition of **M**. The circle denotes a π-rotation on the second mode, i.e. a unitary of the form $R_2 = \exp\left\{\mathrm{i}\pi a_2^\dagger a_2\right\}$. The boxes correspond to $SU(2)$ rotations, i.e. to evolution operators of the form $B_{jk}(\theta_{jk}) = \exp\left\{-\mathrm{i}\theta_{jk}\left(a_j a_k^\dagger + a_k a_j^\dagger\right)\right\}$, corresponding to the transformations

$$B_{jk}^\dagger(\theta_{jk}) \begin{pmatrix} a_j \\ a_k \end{pmatrix} B_{jk}(\theta_{jk}) = \begin{pmatrix} \cos\theta_{ij} & \sin\theta_{ij} \\ -\sin\theta_{ij} & \cos\theta_{ij} \end{pmatrix} \begin{pmatrix} a_j \\ a_k \end{pmatrix}. \tag{16}$$

By an explicit construction, we have

$$U_\gamma = [\mathbb{I}_1 \otimes R_2 \otimes \mathbb{I}_3][B_{23}(\theta_{23}) \otimes \mathbb{I}_3][B_{13}(\theta_{13}) \otimes \mathbb{I}_2][B_{12}(\theta_{12}) \otimes \mathbb{I}_1],$$

[1] Actually, a solution involving a $SU(1,1)$ interaction between a_2 and a_3 followed by an $SU(2)$ interaction between a_1 and a_2 may be found for $|\gamma| > 1$ and then extended to the whole range of $|\gamma|$ by rescaling. However, this solution unavoidably introduces a larger amount of noise compared to that of equations (28) and (29) and it will not be considered here.

where

$$\cos\theta_{23} = \sqrt{\frac{1+\gamma^2}{2}}, \qquad \cos\theta_{13} = \sqrt{\frac{2\gamma^2}{1+\gamma^2}}, \qquad \cos\theta_{12} = \sqrt{\frac{\gamma^2}{1+\gamma^2}}. \tag{17}$$

Other decompositions may also be found, allowing for permutations of modes and different rotations. For $\gamma \to 1$, the mode a_3 decouples from the other two modes and the scheme reduces to the joint measurement of quadratures for the normal operator Z_1 [7].

Each outcome from the joint measurement of the quadratures Q_1 and P_2 corresponds to a complex number $\tau = Q_1 + \mathrm{i}P_2$ that represents a realization of the observable T_γ. The probability density of the outcomes $K_\gamma(\tau)$ for a given initial preparation $R \otimes \sigma$ is obtained as the Fourier transform of the moment generating function $\Xi(\lambda)$:

$$K_\gamma(\tau) = \int \frac{\mathrm{d}^2\lambda}{\pi^2}\, \mathrm{e}^{\lambda^*\tau - \lambda\tau^*}\, \Xi(\lambda), \tag{18}$$

where

$$\Xi(\lambda) = \mathrm{Tr}\big[R \otimes \sigma\, \mathrm{e}^{\lambda T_\gamma^\dagger - \lambda^* T_\gamma}\big]. \tag{19}$$

Using equations (15) we have $\exp\big\{\lambda T_\gamma^\dagger - \lambda^* T_\gamma\big\} = D_1(\lambda) \otimes D_2(-\lambda\gamma) \otimes D_3(-\lambda\kappa)$, where $D_j(z)$ is the displacement operator for the mode a_j. Therefore, the moment generating function can be rewritten as

$$\Xi_\gamma(\lambda) = \chi_{12}(\lambda)\chi_3(-\lambda\kappa), \tag{20}$$

where $\chi_{12}(\lambda) = \mathrm{tr}\,[R D_1(\lambda) \otimes D_2(-\lambda\gamma)]$ and $\chi_3(z) = \mathrm{Tr}[\sigma D_3(z)]$ is the characteristic function of the mode a_3. Using (20) it is easy to see that the probability density of the outcomes is given by the convolution

$$K_\gamma(\tau) = \frac{1}{\kappa^2} H_\gamma(\tau) \star W_3(-\tau/\kappa), \tag{21}$$

with $W_3(z)$ being the Wigner function of the mode a_3, $\star$ the convolution product and $H_\gamma(z)$ the density obtained by the Fourier transform of $\chi_{12}(\lambda)$. In turn, for factorized preparations $R = \varrho_1 \otimes \varrho_2$, the moment generating function $\chi_{12}(\lambda) = \chi_1(\lambda)\chi_2(-\lambda\gamma)$ factorizes into the product of the characteristic functions of ϱ_1 and ϱ_2, respectively, and the density $H_\gamma(\tau)$ reduces to the convolution of the Wigner functions of the two input signals

$$H_\gamma(\tau) = \frac{1}{\gamma^2} W_1(\tau) \star W_2(-\tau/\gamma). \tag{22}$$

Using (15) it is straightforward to see how the variances of the measured quantities Q_1 and P_2 are related to the variances of the quadratures of interest. We have

$$\Delta Q_1^2 = \Delta X_\gamma^2 + \tfrac{1}{2}(1-\gamma^2)\Delta q_3^2 \qquad \Delta P_2^2 = \Delta Y_\gamma^2 + \tfrac{1}{2}(1-\gamma^2)\Delta p_3^2, \tag{23}$$

where $\Delta q_3^2 = \mathrm{Tr}\big[\sigma q_3^2\big]$ and analogously $\Delta p_3^2 = \mathrm{Tr}\big[\sigma p_3^2\big]$ (remind that equation (6) implies $\mathrm{Tr}[\sigma q_3] = \mathrm{Tr}[\sigma p_3] = 0$). Note that the added noise in equation (23) is the minimum noise according to generalized uncertainty relations for joint measurement of non-commuting observables [1–5]. On the other hand, the covariance between the measured quadratures, i.e. the quantity

$$\Sigma_{Q_1P_2} = \tfrac{1}{2}\mathrm{Tr}_{12a}[R \otimes \sigma(Q_1P_2 + P_2Q_1)] - \mathrm{Tr}_{12a}[R \otimes \sigma Q_1]\mathrm{Tr}_{12a}[R \otimes \sigma P_2] \tag{24}$$

may be written as

$$\Sigma_{Q_1P_2} = \Sigma_{X_\gamma Y_\gamma} - \tfrac{1}{2}(1-\gamma^2)\mathrm{Tr}_a\big[\tfrac{1}{2}\sigma(p_3q_3 + q_3p_3)\big], \tag{25}$$

where $\Sigma_{X_\gamma Y_\gamma} = \frac{1}{2}\mathrm{Tr}_{12}[R(X_\gamma Y_\gamma + Y_\gamma X_\gamma)] - \mathrm{Tr}_{12}[RX_\gamma]\,\mathrm{Tr}_{12}[RY_\gamma]$ is the covariance of the desired quadratures.

Note that the added noise to the covariance, equation (25), may vanish for some preparation of the state σ whereas the added noise to the variances, equation (23), cannot vanish for any physical preparation σ. This raises the question of the consequences of different field states on the statistics of the measurement and, in turn, of the role played by preparations of states in concrete experiments. On the other hand, within experimental frameworks, one may take full advantage of possible freedom in preparing some of the modes. This is definitively the case of the Naimark mode a_3, even though its preparation needs to be compatible with the prescription (6) for the expectation values of position and momentum operators. In particular, a valid Naimark extension can be obtained by preparing the mode a_3 in the vacuum state $\sigma = |0\rangle\langle 0|$ to let its contribution to the noise in formula (25) to vanish, since $\mathrm{Tr}_a[\sigma(q_3 p_3 + p_3 q_3)] = 0$, and to minimize Δq_3^2 and Δp_3^2 in (23), since both the terms would be equal to $1/2$. Each of the other two fields may be, for instance, in one among the most meaningful types of states, such as number states, coherent states, thermal states or phase states (i.e. eigenstates of the operator $C + \mathrm{i}S$, where C and S are 'cosine' and 'sine' operators, respectively) or prepared in an entangled states. If we consider the fully separable state described by the density operator $\varrho = R \otimes \sigma = \varrho_1 \otimes \varrho_2 \otimes \sigma$, where ϱ_k, with $k = 1, 2$, denotes the preparation for the kth bosonic field in the arbitrarily mixed state $\varrho_k = \sum_{m=0}^{\infty} p_m^{(k)} |m\rangle\langle m|$ on the Hilbert space $\mathcal{H}_k$, then the system moment generating function is easily obtained by resorting to

$$\mathrm{Tr}_k[\varrho_k D_k(\alpha_k)] = \mathrm{e}^{-\frac{|\alpha_k|^2}{2}} \sum_{m=0}^{\infty} p_m^{(k)} L_m(|\alpha_k|^2), \tag{26}$$

where the L_n's are Laguerre polynomials. For instance, for coherent and phase states equation (26) should be used with

$$p_m^{(k)} = \mathrm{e}^{-|\alpha|^2} \frac{|\alpha|^{2m}}{m!} \qquad \text{and} \qquad p_m^{(k)} = (1 - |z|^2)|z|^{2m}, \tag{27}$$

respectively (phase state formulae can be used even when dealing with thermal states upon the identification $z = \exp\left[-\frac{1}{2}\beta\hbar\omega\right]$, β being the inverse of temperature). Suppose no specific conditions do constraint, in principle, the preparation for the mode a_2. Once again a vacuum choice may be advantageous in some respects. Let us therefore focus on the specific case of the measurement of Z_γ on the class of factorized signals described by $R = \varrho_1 \otimes |0\rangle\langle 0|$, where ϱ_1 is a generic preparation of the mode a_1 while $|0\rangle$ is the ground state of the mode a_2. In this case, $\varrho = \varrho_1 \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|$, equation (21) becomes a Gaussian convolution and the moment generating function becomes independent of the parameter γ:

$$\Xi(\lambda) = \chi_1(\lambda) \exp\left(-\tfrac{1}{2}|\lambda|^2\right). \tag{28}$$

The measured variances are thus given by

$$\Delta Q_1^2 = \tfrac{1}{2}\left(\Delta q_1^2 + 1\right) \qquad \Delta P_2^2 = \tfrac{1}{2}\left(\Delta p_1^2 + 1\right). \tag{29}$$

Equations (28) and (29) contain a *remarkable result* that may be expressed as follows. The measurement of Z_γ on the class of states $R = \varrho_1 \otimes |0\rangle\langle 0|$ *does not* lead to added noise with respect to the measurement of the normal operator Z_1.

This result finds a natural application in the context of *heterodyne detection*, where currents of the form (1) show up. As is known, in heterodyne detection a single-mode signal field E_1 of nominal frequency ω_1 is mixed through a beam splitter with a local oscillator field E_L whose frequency ω_L is slightly offset by an amount $\omega_I \ll \omega_1$ from that of the input signal, i.e. $\omega_1 = \omega_L + \omega_I$. A photodetector is placed right after the beam splitter (see figure 2). The output photocurrent, which generally depends on fields parameters and on specific assumptions on the apparatus, is filtered at the *intermediate frequency* ω_I.

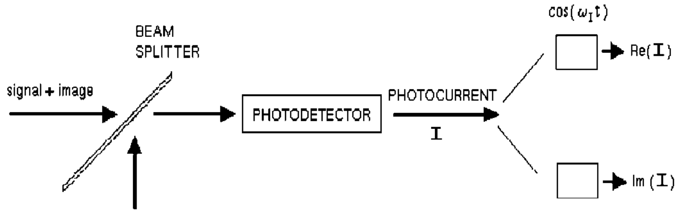

Figure 2. The scheme for heterodyne detection.

In standard optical heterodyne detection (see e.g. [8]), measuring the filtered photocurrent corresponds to realize the quantum measurement of the normal operator $y = a_1 + a_2^\dagger$ [8], where a_1 (resp. $a_2^\dagger$) denotes the photon annihilator (resp. creation) operator for the input (resp. image) signal. Measuring the real and imaginary parts of the (actually rescaled) output photocurrent thus provides the simultaneous measurement of both input field quadratures. Nevertheless, it has been also argued that whenever one is not restricted to an input field frequency in the optical regime, but, rather, one is concerned with microwave (or radio) heterodyning, then the interaction of the input signal field with the apparatus of figure 2 (approximatively) results in the measurement operator

$$y_C = \sqrt{\left(1 + \frac{\omega_I}{\omega_1}\right)}\, a_1 + \sqrt{\left(1 - \frac{\omega_I}{\omega_1}\right)}\, a_2^\dagger$$

(see [9] and discussion in [8]).

Since $\left[y_C, y_C^\dagger\right] = 2\frac{\omega_I}{\omega_1} \neq 0$, the Caves measurement operator y_C is not compatible with simultaneous measurements of signal quadratures. In other words, standard heterodyne detection cannot achieve the measurement of the Caves operator and a question arises on whether simultaneous phase and amplitude measurements may be accomplished in this case. The answer may be found in the results reported above. In fact, the measurement of the Caves operator corresponds to the generalized measurement of the non-normal operator

$$Z_{\gamma_C} = a_1 + \gamma_C a_2^\dagger, \qquad \gamma_C = \sqrt{\frac{\omega_1 - \omega_I}{\omega_1 + \omega_I}} < 1 \quad . \tag{30}$$

In the light of our previous results, we thus learn that the simultaneous measurement of the field quadratures for a quasi-monochromatic signal can be realized even in the case when the heterodyne apparatus yields a measurement operator of the Caves-type, equation (30). To this aim, it suffices to generalize the heterodyne detection scheme by introducing a single-boson Naimark mode and letting it interact with the other modes through the linear transformation (9). Moreover, a suitable preparation enables one to avoid additional noise with respect to that resulting in the measurement of signal field quadratures within the framework of the standard optical heterodyne detection.

It is worth also discussing the matter from the point of view of phase operators since our results can be used to proceed in defining a *feasible phase* within the Caves description of heterodyning. Since the operator T is normal, then its associated self-adjoint phase operator

$$\theta_T = \frac{1}{2i} \ln \frac{T}{T^\dagger} \tag{31}$$

can be defined unambiguously indeed so that cosine and sine quadrature operators

$$C = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\theta_T} + \mathrm{e}^{-\mathrm{i}\theta_T}), \qquad S = \frac{1}{2i}(\mathrm{e}^{\mathrm{i}\theta_T} - \mathrm{e}^{-\mathrm{i}\theta_T})$$

obey the correct relation $C^2+S^2 = 1$. It is now in order to recalling that the two-modes relative number state representation discussed by Ban (see [11] and references therein) fits fairly with the feasible phase concept of Shapiro and Wagner (namely, the shift-phase operator associated with the Shapiro–Wagner measurement operator $y = a_1 + a_2^\dagger$). Upon defining the three-mode relative number operator $N = N_1 - N_2 - N_3$, where $N_k = a_k^\dagger a_k$ $(k = 1, 2, 3)$, one gets

$$[\mathrm{e}^{\mathrm{i}\theta_T}, N] = \mathrm{e}^{\mathrm{i}\theta_T}, \qquad [N, \theta_T] = i. \tag{32}$$

These relations are what one expects for genuine phase operators. In other words, a feasible phase can be naturally defined even in the Caves description of heterodyning at the cost of introducing of a Naimark mode and generalizing the two-modes relative state representation to a three-modes one. The commutator $[N, \theta_T]$ can then be interpreted as the canonical conjugation of the feasible phase for Caves heterodyne measurement operator with respect to the operator mode number difference N.

As final comments, note that tracing out the Naimark mode a_3, and introducing symmetric ordering when needed in equations (31)–(32), formulae given in [10] are recovered. Further, it would be of interest to move towards the direction of generalizing the relative number state representation for the description of the phase operator of the generalized heterodyne measurement we have introduced in this communication, and more generally for operators describing linear amplifiers involving more than three modes. This is also concerned with the investigation of the possibility of extracting basic algebraic structures underlying these systems to generalize algebras given in [10]. These issues are currently under investigation and results will be reported elsewhere.

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